1. RELATIONS

In class 11 we discussed about the Cartesian product of sets. Now, we extend our ideas to relation in a set and then in next chapter we shall be taking up functions.

RELATION IN A SET

A relation R in a set A is a subset of $A \times A$.

Thus, R is a relation in a set $A \Leftrightarrow R \subseteq A \times A$.

If $(a, b) \in R$ *then we say that a is related to b and write, a R b.*

If $(a, b) \notin R$ *then we say that a is not related to b and write, a* $\not R$ *b.*

Example Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be a relation in A, given by
$R = \{(a, b) : a - b = 2\}$.
Then, $R = \{(3, 1), (4, 2), (5, 3), (6, 4)\}$.
Clearly, $3\ R\ 1$, $4\ R\ 2$, $5\ R\ 3$ and $6\ R\ 4$.
But, $1 \not R\ 3$, $2 \not R\ 4$, $5 \not R\ 6$, etc.

DOMAIN AND RANGE OF A RELATION

Let R be a relation in a set A. Then, the set of all first coordinates of elements of R is called the domain of R, written as dom (R) and the set of all second coordinates of R is called the range of R, written as range (R).

$\therefore$ dom $(R) = \{a : (a, b) \in R\}$ and range $(R) = \{b : (a, b) \in R\}$.

Example Let $A = \{1, 2, 3, 4, \ldots, 15, 16\}$ and let R be a relation in A, given by $R = \{(a, b) : b = a^2\}$.

Then, $R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$.
$\therefore$ dom $(R) = \{1, 2, 3, 4\}$ and range $(R) = \{1, 4, 9, 16\}$.

Some Particular Types of Relations

EMPTY RELATION (Or VOID RELATION) *A relation R in a set A is called an empty relation, if no element of A is related to any element of A and we denote such a relation by* ϕ.

Thus, $R = \phi \subseteq A \times A$.

Example Let $A = \{1, 2, 3, 4, 5\}$ and let R be a relation in A, given by $R = \{(a, b) : a - b = 6\}$.

Clearly, no element $(a, b) \in A \times A$ satisfies the property $a - b = 6$.

$\therefore$ R is an empty relation in A.

UNIVERSAL RELATION *A relation R in a set A is called a universal relation, if each element of A is related to every element of A.*

Thus, $R = (A \times A) \subseteq (A \times A)$ is the universal relation on A.

Example Let $A = \{1, 2, 3\}$. Then,
$R = (A \times A) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
is the universal relation in A.

IDENTITY RELATION *The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.*

Example Let $A = \{1, 2, 3\}$. Then,
$I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

VARIOUS TYPES OF RELATIONS

Let A be a nonempty set. Then, a relation R on A is said to be

(i) **reflexive** if $(a, a) \in R$ for each $a \in A$,
i.e., if $a \; R \; a$ for each $a \in A$.

(ii) **symmetric** if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$,
i.e., if $a \; R \; b \Rightarrow b R a$ for all $a, b \in A$.

(iii) **transitive** if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$,
i.e., if $a \; R \; b$ and $b \; R \; c \Rightarrow a \; R \; c$.

EQUIVALENCE RELATION *A relation R in a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.*

SOLVED EXAMPLES

EXAMPLE 1 *Let A be the set of all triangles in a plane and let R be a relation in A, defined by $R = \{(\triangle_1, \triangle_2) : \triangle_1 \cong \triangle_2\}$.*
Show that R is an equivalence relation in A.

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary triangle in A. Then,

$\triangle \cong \triangle \Rightarrow (\triangle, \triangle) \in R$ for all values of $\triangle$ in A.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $\triangle_1, \triangle_2 \in A$ such that $(\triangle_1, \triangle_2) \in R$. Then,

$$(\triangle_1, \triangle_2) \in R \Rightarrow \triangle_1 \cong \triangle_2$$
$$\Rightarrow \triangle_2 \cong \triangle_1$$
$$\Rightarrow (\triangle_2, \triangle_1) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $\triangle_1, \triangle_2, \triangle_3 \in A$ such that $(\triangle_1, \triangle_2) \in R$ and $(\triangle_2, \triangle_3) \in R$. Then, $(\triangle_1, \triangle_2) \in R$ and $(\triangle_2, \triangle_3) \in R$

$\Rightarrow \triangle_1 \cong \triangle_2$ and $\triangle_2 \cong \triangle_3$

$\Rightarrow \triangle_1 \cong \triangle_2$

$\Rightarrow (\triangle_1, \triangle_3) \in R.$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

EXAMPLE 2 *Let A be the set of all lines in xy-plane and let R be a relation in A, defined by*

$$R = \{(L_1, L_2) : L_1 \,||\, L_2\}.$$

Show that R is an equivalence relation in A.
Find the set of all lines related to the line $y = 3x + 5$.

SOLUTION The given relation satisfies the following properties:

(i) *Reflexivity*

Let L be an arbitrary line in A. Then,

$$L \,||\, L \Rightarrow (L, L) \in R \;\forall\; L \in A.$$

Thus, R is reflexive.

(ii) *Symmetry*

Let $L_1, L_2 \in A$ such that $(L_1, L_2) \in R$. Then,

$$(L_1, L_2) \in R \Rightarrow L_1 \,||\, L_2$$
$$\Rightarrow L_2 \,||\, L_1$$
$$\Rightarrow (L_2, L_1) \in R.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $L_1, L_2, L_3 \in A$ such that $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$. Then, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

$\Rightarrow L_1 \,||\, L_2$ and $L_2 \,||\, L_3$

$\Rightarrow\ L_1 \,||\, L_3$

$\Rightarrow\ (L_1, L_3) \in R.$

$\therefore$ R is transitive.

Thus R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

The family of lines parallel to the line $y = 3x + 5$ is given by $y = 3x + k$, where k is real.

EXAMPLE 3 *Let Z be the set of all integers and let R be a relation in Z, defined by $R = \{(a, b) : (a - b)$ is even$\}$. Show that R is an equivalence relation in Z.*

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let a be an arbitrary element of Z.

Then, $(a - a) = 0$, which is even.

$\therefore\ (a, a) \in R\ \forall\ a \in Z.$

So, R is reflexive.

(ii) *Symmetry*
Let $a, b \in Z$ such that $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is even

$\Rightarrow -(a - b)$ is even

$\Rightarrow (b - a)$ is even

$\Rightarrow (b, a) \in R.$

$\therefore$ R is symmetric.

(iii) *Transitivity*
Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a - b)$ is even and $(b - c)$ is even

$\Rightarrow \{(a - b) + (b - c)\}$ is even

$\Rightarrow (a - c)$ is even

$\Rightarrow (a, c) \in R.$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation in Z.

EXAMPLE 4 *Let A be the set of all lines in a plane and let R be a relation in A defined by*

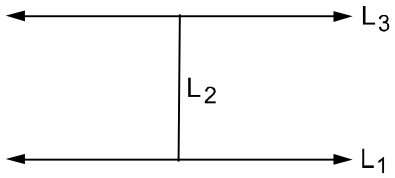
$$R = \{(L_1, L_2) : L_1 \perp L_2\}.$$

Show that R is symmetric but neither reflexive nor transitive.

SOLUTION Clearly, any line L cannot be perpendicular to itself.

$\therefore$ $(L, L) \notin R$ for any $L \in A$.

So, R is not reflexive.



Again, let $(L_1, L_2) \in R$. Then,

$$(L_1, L_2) \in R \Rightarrow L_1 \perp L_2$$
$$\Rightarrow L_2 \perp L_1$$
$$\Rightarrow (L_2, L_1) \in R.$$

$\therefore$ R is symmetric.

Now, let $L_1, L_2, L_3 \in A$ such that $L_1 \perp L_2$ and $L_2 \perp L_3$.

Then, clearly L_1 is not perpendicular to L_3.

Thus, $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$, but $(L_1, L_3) \notin R$.

$\therefore$ R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

EXAMPLE 5 *Let S be the set of all real numbers and let R be a relation in S defined by*

$$R = \{(a, b) : (1 + ab) > 0\}.$$

Show that R is reflexive and symmetric but not transitive.

SOLUTION Let a be any real number. Then,

$$(1 + aa) = (1 + a^2) > 0 \text{ shows that } (a, a) \in R \ \forall \ a \in S.$$

$\therefore$ R is reflexive.

Also, $(a, b) \in R \Rightarrow (1 + ab) > 0$

$$\Rightarrow (1 + ba) > 0 \ [\because \ ab = ba]$$
$$\Rightarrow (b, a) \in R.$$

$\therefore$ R is symmetric.

In order to show that R is not transitive, consider $(-1, 0)$ and $(0, 2)$.

Clearly, $(-1, 0) \in R$, since $[1 + (-1) \times 0] > 0$.

And, $(0, 2) \in R$, since $[1 + 0 \times 2] > 0$.

But, $(-1, 2) \notin R$, since $[1 + (-1) \times 2]$ is not greater than 0.

Hence, R is reflexive and symmetric but not transitive.

EXAMPLE 6 *Let S be the set of all real numbers and let R be a relation in S, defined by*

$$R = \{(a, b) : a \leq b\}.$$

Show that R is reflexive and transitive but not symmetric.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary real number.

Then, $a \leq a \Rightarrow (a, a) \in R$.

Thus, $(a, a) \in R \ \forall \ a \in S$.

$\therefore$ R is reflexive.

(ii) *Transitivity*

Let a, b, c be real numbers such that $(a, b) \in R$ and $(b, c) \in R$.

Then, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow a \leq b$ and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$.

$\therefore$ R is transitive.

(iii) *Nonsymmetry*

Clearly, $(4, 5) \in R$ since $4 \leq 5$.

But, $(5, 4) \notin R$ since $5 \leq 4$ is not true.

$\therefore$ R is not symmetric.

EXAMPLE 7 *Let S be the set of all real numbers and let R be a relation in S, defined by* $R = \{(a, b) : a \leq b^2\}$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

SOLUTION (i) *Nonreflexivity*

Clearly, $\frac{1}{2}$ is a real number and $\frac{1}{2} \leq \left(\frac{1}{2}\right)^2$ is not true.

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right) \notin R$.

Hence, R is not reflexive.

(ii) *Nonsymmetry*

Consider the real numbers $\frac{1}{2}$ and 1.

Clearly, $\frac{1}{2} \leq 1^2 \Rightarrow \left(\frac{1}{2}, 1\right) \in R$.

But, $1 \leq \left(\frac{1}{2}\right)^2$ is not true and so $\left(1, \frac{1}{2}\right) \notin R$.

Thus, $\left(\frac{1}{2}, 1\right) \in R$ but $\left(1, \frac{1}{2}\right) \in R$.

Hence, R is not symmetric.

(iii) *Nontransitivity*

Consider the real numbers 2, –2 and 1.

Clearly, $2 \le (-2)^2$ and $-2 \le (1)^2$ but $2 \le 1^2$ is not true.

Thus, $(2, -2) \in R$ and $(-2, 1) \in R$, but $(2, 1) \notin R$.

Hence, R is not transitive.

EXAMPLE 8 *Let S be the set of all real numbers and let R be a relation in S, defined by* $R = \{(a, b) : a \le b^3\}$.

Show that R satisfies none of reflexivity, symmetry and transitivity.

SOLUTION (i) *Nonreflexivity*

Clearly, $\frac{1}{2}$ is a real number and $\frac{1}{2} \le \left(\frac{1}{2}\right)^3$ is not true.

$\therefore \quad \left(\frac{1}{2}, \frac{1}{2}\right) \notin R.$

Hence, R is not reflexive.

(ii) *Nonsymmetry*

Take the real numbers $\frac{1}{2}$ and 1.

Clearly, $\frac{1}{2} \le 1^3$ is true and therefore, $\left(\frac{1}{2}, 1\right) \in R.$

But, $1 \le \left(\frac{1}{2}\right)^3$ is not true and so $\left(1, \frac{1}{2}\right) \notin R.$

Hence, R is not symmetric.

(iii) *Nontransitivity*

Consider the real numbers 3, $\frac{3}{2}$ and $\frac{4}{3}$.

Clearly, $3 \le \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} \le \left(\frac{4}{3}\right)^3$ but $3 \le \left(\frac{4}{3}\right)^3$ is not true.

Thus, $\left(3, \frac{3}{2}\right) \in R$ and $\left(\frac{3}{2}, \frac{4}{3}\right) \in R$, but $\left(3, \frac{4}{3}\right) \notin R.$

Hence, R is not transitive.

Thus, R satisfies none of reflexivity, symmetry and transitivity.

EXAMPLE 9 *Let N be the set of all natural numbers and let R be a relation in N, defined by*

$$R = \{(a, b)\} : a \text{ is a factor of } b\}.$$

Then, show that R is reflexive and transitive but not symmetric.

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary element of N.

Then, clearly, a is a factor of a.

$\therefore \ (a, \ a) \in R \ \forall \ a \in N$.

So, R is reflexive.

(ii) *Transitivity*

Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ (a is a factor of b) and (b is a factor of c)

$\Rightarrow$ $b = ad$ and $c = be$ for some $d, e \in N$

$\Rightarrow$ $c = (ad)\ e = a(de)$ [by associative law]

$\Rightarrow$ a is a factor of c

$\Rightarrow$ $(a, c) \in R$.

$\therefore \ (a, b) \in R$ and $(b, c) \in R \ \Rightarrow \ (a, c) \in R$.

Hence, R is transitive.

(iii) *Nonsymmetry*

Clearly, 2 and 6 are natural numbers and 2 is a factor of 6.

$\therefore \ (2, 6) \in R$.

But, 6 is not a factor of 2.

$\therefore \ (6, 2) \notin R$.

Thus, $(2, 6) \in R$ and $(6, 2) \notin R$.

Hence, R is not symmetric.

EXAMPLE 10 *Let N be the set of all natural numbers and let R be a relation in N, defined by*

$R = \{(a, \ b) : a \text{ is a multiple of } b\}$.

Show that R is reflexive and transitive but not symmetric.

SOLUTION Here R satisfies the following properties:

(i) *Reflexivity*

Let a be an arbitrary element of N.

Then, $a = (a \times 1)$ shows that a is a multiple of a.

$\therefore \ (a, a) \in R \ \forall \ a \in N$.

So, R is reflexive.

(ii) *Transitivity*

Let $a, b, c \in N$ such that $(a, b) \in R$ and $(b, c) \in R$.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ (a is a multiple of b) and (b is a multiple of c)

$\Rightarrow$ $a = bd$ and $b = ce$ for some $d \in N$ and $e \in N$

$\Rightarrow$ $a = (ce)d$

$\Rightarrow$ $a = c(ed)$

$\Rightarrow$ a is a multiple of c

$\Rightarrow$ $(a, c) \in R$.

$\therefore$ $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Hence, R is transitive.

(iii) *Nonsymmetry*

Clearly, 6 and 2 are natural numbers and 6 is a multiple of 2.

$\therefore$ $(6, 2) \in R$.

But, 2 is not a multiple of 6.

$\therefore$ $(2, 6) \notin R$.

Thus, $(6, 2) \in R$ and $(2, 6) \notin R$.

Hence, R is not symmetric.

EXAMPLE 11 *Let X be a nonempty set and let S be the collection of all subsets of X. Let R be a relation in S, defined by*

$R = \{(A, B) : A \subset B\}$.

Show that R is transitive but neither reflexive nor symmetric.

SOLUTION Clearly, R satisfies the following properties:

(i) *Transitivity*

Let $A, B, C \in S$ such that $(A, B) \in R$ and $(B, C) \in R$.

Now, $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow$ $A \subset B$ and $B \subset C$

$\Rightarrow$ $A \subset C$

$\Rightarrow$ $(A, C) \in R$.

$\therefore$ R is transitive.

(ii) *Nonreflexivity*

Let A be any set in S.

Then, $A \not\subset A$ shows that $(A, A) \notin R$.

$\therefore$ R is not reflexive.

(iii) *Nonsymmetry*

Now $(A, B) \in R \Rightarrow A \subset B$

$\Rightarrow B \not\subset A$

$\Rightarrow (B, A) \notin R$.

$\therefore$ R is not symmetric.

Hence, R is transitive but neither reflexive nor symmetric.

EXAMPLE 12 *Give an example of a relation which is*

(i) reflexive and transitive but not symmetric;
(ii) symmetric and transitive but not reflexive;
(iii) reflexive and symmetric but not transitive;
(iv) symmetric but neither reflexive nor transitive;
(v) transitive but neither reflexive nor symmetric.

SOLUTION Let $A = \{1, 2, 3\}$.

Then, it is easy to verify that the relation

(i) $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$

is reflexive and transitive.

R_1 is not symmetric, since

$(1, 2) \in R$ and $(2, 1) \notin R$.

(ii) $R_2 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$

is symmetric and transitive.

But, R_2 is not reflexive, since $(3, 3) \notin R_2$.

(iii) $R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$

is reflexive and symmetric.

But, R_3 is not transitive, since

$(1, 2) \in R_3$, $(2, 3) \in R_3$ but $(1, 3) \notin R_3$.

(iv) $R_4 = \{(2, 2), (3, 3), (1, 2), (2, 1)\}$

is symmetric.

But, R_4 is not reflexive since $(1, 1) \notin R_4$.

Also, R_4 is not transitive, as

$(1, 2) \in R_4$ and $(2, 1) \in R_4$ but $(1, 1) \in R_4$.

(v) $R_5 = \{(2, 2), (3, 3), (1, 2)\}$

is transitive.

But, R_5 is not reflexive, since $(1, 1) \notin R$.

And, R_5 is not symmetric as $(1, 2) \in R_5$ but $(2, 1) \notin R_5$.

EXAMPLE 13 *Let N be the set of all natural numbers and let R be a relation on $N \times N$, defined by*

$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$.

Show that R is an equivalence relation.

SOLUTION Here R satisfies the following properties:

(i) *Reflexivity*

Let $(a, b) \Rightarrow R$. Then,

$(a, b)\ R\ (a, b)$, since $ab = ba$

[by commutative law of multiplication on N].

Thus, $(a, b)\ R\ (a, b)\ \forall\ (a, b) \in R$.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $(a, b)\ R\ (c, d)$. Then,

$$(a, b)\ R\ (c, d) \Rightarrow ad = bc$$
$$\Rightarrow bc = ad$$
$$\Rightarrow cb = da$$

[by commutativity of multiplication on N]

$$\Rightarrow (c, d)\ R\ (a, b).$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$ad = bc$ and $cf = de$

$\Rightarrow adcf = bcde$

$\Rightarrow (af)(cd) = (be)(cd)$

$\Rightarrow af = be$ [by cancellation law]

$\Rightarrow (a, b)\ R\ (e, f)$.

$\therefore$ $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$.

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

EXAMPLE 14 *If R_1 and R_2 be two equivalence relations on a set A, prove that $R_1 \cap R_2$ is also an equivalence relation on A.*

SOLUTION Let R_1 and R_2 be two equivalence relations on a set A.

Then, $R_1 \subseteq A \times A$, $R_2 \subseteq A \times A \Rightarrow (R_1 \cap R_2) \subseteq A \times A$.

So, $(R_1 \cap R_2)$ is a relation on A.

This relation on A satisfies the following properties.

(i) *Reflexivity*

R_1 is reflexive and R_2 is reflexive

$\Rightarrow (a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$

$\Rightarrow (a, a) \in R_1 \cap R_2$ for all $a \in A$

$\Rightarrow R_1 \cap R_2$ is reflexive.

(ii) *Symmetry*

Let (a, b) be an arbitrary element of $R_1 \cap R_2$. Then,

$(a, b) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2$

[$\because$ R_1 is symmetric and R_2 is symmetric]

$\Rightarrow (b, a) \in R_1 \cap R_2$.

This shows that $R_1 \cap R_2$ is symmetric.

(iii) *Transitivity*

$(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$, $(a, b) \in R_2$, and $(b, c) \in R_1$, $(b, c) \in R_2$

$\Rightarrow \{(a, b) \in R_1, (b, c) \in R_1\}$, and $\{(a, b) \in R_2, (b, c) \in R_2\}$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2$

[$\because$ R_1 is transitive and R_2 is transitive]

$\Rightarrow (a, c) \in R_1 \cap R_2$.

This shows that $(R_1 \cap R_2)$ is transitive.

Thus, $R_1 \cap R_2$ is reflexive, symmetric and transitive.

Hence, $R_1 \cap R_2$ is an equivalence relation.

EXAMPLE 15 *Give an example to show that the union of two equivalence relations on a set A need not be an equivalence relation on A.*

SOLUTION Let R_1 and R_2 be two relations on a set $A = \{1, 2, 3\}$, given by

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$.

Then, it is easy to verify that each one of R_1 and R_2 is an equivalence relation.

But, $R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$
is not transitive, as

$(3, 1) \in R_1 \cup R_2$ and $(1, 2) \in R_1 \cup R_2$ but $(3, 2) \notin R_1 \cup R_2$.

Hence, $(R_1 \cup R_2)$ is not an equivalence relation.

EQUIVALENCE CLASSES *Let R be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called the equivalence class determined by a and it is denoted by* $[a]$.

Thus, $[a] = \{b \in A : (a, b) \in R\}$.

Two equivalence classes are either disjoint or identical.

An Important Result *An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.*

EXAMPLE 16 *On the set Z of all integers, consider the relation*
$R = \{(a, b) : (a - b)$ *is divisible by* $3\}$.
Show that R is an equivalence relation on Z.
Also find the partitioning of Z into mutually disjoint equivalence classes.

SOLUTION The relation R on Z satisfies the following properties:

(i) *Reflexivity*

Let $a \in Z$.

Then, $(a - a) = 0$, which is divisible by 3.

$\therefore \quad a \, R \, a \; \forall \; a \in Z$.

So, R is reflexive.

(ii) *Symmetry*

Let $a, b \in Z$ such that $a \, R \, b$. Then,

$$\begin{aligned} a \, R \, b &\Rightarrow (a - b) \text{ is divisible by } 3 \\ &\Rightarrow -(a - b) \text{ is divisible by } 3 \\ &\Rightarrow (b - a) \text{ is divisible by } 3 \\ &\Rightarrow b \, R \, a. \end{aligned}$$

$\therefore \quad a \, R \, b \Rightarrow b \, R \, a \; \forall \; a, b \in Z$.

So, R is symmetric.

(iii) *Transitivity*

Let $a, b, c \in Z$ such that $a \, R \, b$ and $b \, R \, c$. Then,

$$\begin{aligned} a \, R \, b, \; b \, R \, c &\Rightarrow (a - b) \text{ is divisible by } 3 \\ &\qquad \text{and } (b - c) \text{ is divisible by } 3 \\ &\Rightarrow [(a - b) + (b - c)] \text{ is divisible by } 3 \\ &\Rightarrow (a - c) \text{ is divisible by } 3. \end{aligned}$$

Thus, $a \, R \, b, \; b \, R \, c \Rightarrow a \, R \, c \; \forall \; a, b, c \in Z$.

$\therefore$ R is an equivalence relation on Z.

Now, let us consider [0], [1] and [2].

We have:

$$[0] = \{x \in Z : x\ R\ 0\}$$
$$= \{x \in Z : (x - 0) \text{ is divisible by } 3\}$$
$$= \{..., -6, -3, 0, 3, 6, 9, ...\}.$$

$\therefore$ $[0] = \{..., -6, -3, 0, 3, 6, 9, ...\}$.

Similarly, $[1] = \{x \in Z : x\ R\ 1\}$

$$= \{x \in Z : (x - 1) \text{ is divisible by } 3\}$$
$$= \{..., -5, -2, 1, 4, 7, 10, ...\}.$$

$\therefore$ $[1] = \{..., -5, -2, 1, 4, 7, 10, ...\}$.

And, $[2] = \{x \in Z : x\ R\ 2\}$

$$= \{x \in Z : (x - 2) \text{ is divisible by } 3\}$$
$$= \{..., -4, -1, 2, 5, 8, 11, ...\}.$$

$\therefore$ $[2] = \{..., -4, -1, 2, 5, 8, 11, ...\}$.

Clearly, [0], [1] and [2] are mutually disjoint

and $Z = [0] \cup [1] \cup [2]$.

EXAMPLE 17 *Let* $A = \{x \in Z : 0 \le x \le 12\}$.
Show that $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$ *is*
(i) reflexive, (ii) symmetric and (iii) transitive.
Find the set of elements related to 1. **[CBSE 2010]**

SOLUTION Clearly, $A = \{0, 1, 2, 3, 4, ..., 10, 11, 12\}$.

Here, R satisfies the following properties.

(i) *Reflexivity*

Let a be an arbitrary element of A. Then,

$a - a = 0$, which is a multiple of 4.

$\therefore$ $a\,R\,a$ for all $a \in A$.

So, R is reflexive.

(ii) *Symmetry*

Let $a\,R\,b$. Then,

$$a\,R\,b \Rightarrow |a - b| \text{ is a multiple of } 4$$
$$\Rightarrow |-(a - b)| \text{ is a multiple of } 4$$
$$\Rightarrow |b - a| \text{ is a multiple of } 4$$
$$\Rightarrow b\ R\ a.$$

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $a\,R\,b$ and $b\,R\,c$. Then,

$a\ R\ b,\ b\ R\ c$

$\Rightarrow$ $|a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4.

Let $|a-b| = 4k_1$ and $|b-c| = 4k_2$. Then,

$$|a-c| = |(a-b)-(b-c)| = |4k_1 - 4k_2|$$
$$= |4(k_1 - k_2)| = 4|k_1 - k_2|, \text{ which is a multiple of 4.}$$

$\therefore$ $a\ R\ b,\ b\ R\ c \Rightarrow a\ R\ c$. So, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

Now, $[1] = \{x \in A : x\,R\,1\}$

$= \{x \in A : |x-1| \text{ is a multiple of 4}\}$

$= \{1, 5, 9\}$.

Hence, the required set is {1, 5, 9}.

EXAMPLE 18 *Let* $A = \{1,2,3,4,5,6,7,8,9\}$ *and* R *be a relation in* $A \times A$, *defined by* $(a, b)\,R\,(c, d) \Leftrightarrow a + d = b + c$ *for all* (a, b) *and* $(c, d) \in A \times A$. *Prove that* R *is an equivalence relation. Also obtain the equivalence class determined by* $(2,5)$. **[CBSE 2014]**

SOLUTION (i) *Reflexivity*

Let $(a, b) \in A \times A$. Then,

$(a, b) \in A \times A \Rightarrow a, b \in A$

$\Rightarrow a + b = b + a$

$\Rightarrow (a, b)\ R\ (a, b)$.

$\therefore$ R is reflexive.

(ii) *Symmetry*

Let $(a, b)\ R\,(c, d)$. Then,

$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$

$\Rightarrow c + b = d + a$

$\Rightarrow (c, d)\ R\ (a, b)$.

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b)\ R\ (c, d)$ and $(c, d)\,R\,(e, f)$. Then,

$(a, b)\ R\ (c, d)$ and $(c, d)\,R\,(e, f)$

$\Rightarrow$ $a + d = b + c$ and $c + f = d + e$

$\Rightarrow$ $a + d + c + f = b + c + d + e$

$\Rightarrow$ $a + f = b + e$

$\Rightarrow$ $(a, b)\ R\ (e, f)$.

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

$[(2, 5)] = \{(a, b) : (2, 5)\ R\ (a, b)\}$

$= \{(a, b) : 2 + b = 5 + a\} = \{(a, b) : b - a = 3\}$

$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

EXERCISE 1A

Very-Short-Answer Questions

1. Find the domain and range of the relation
   $R = \{(-1, 1), (1, 1), (-2, 4), (2, 4)\}$.
2. Let $R = \{(a, a^3) : a$ is a prime number less than $5\}$.
   Find the range of R. **[CBSE 2014]**
3. Let $R = \{(a, a^3) : a$ is a prime number less than $10\}$.
   Find (i) R (ii) dom (R) (iii) range (R).
4. Let $R = \{x, y) : x + 2y = 8\}$ be a relation on N.
   Write the range of R. **[CBSE 2014]**
5. Let $R = \{(a, b) : a, b \in N$ and $a + 3b = 12\}$.
   Find the domain and range of R.
6. Let $R = \{(a, b) : b = |a - 1|, a \in Z$ and $|a| < 3\}$.
   Find the domain and range of R.
7. Let $R = \left\{\left(a, \frac{1}{a}\right) : a \in N$ and $1 < a < 5\right\}$.
   Find the domain and range of R.
8. Let $R = \{(a, b) : a, b \in N$ and $b = a + 5, a < 4\}$.
   Find the domain and range of R.
9. Let S be the set of all sets and let $R = \{(A, B) : A \subset B)\}$, i.e., A is a proper subset of B. Show that R is (i) transitive (ii) not reflexive (iii) not symmetric.
10. Let A be the set of all points in a plane and let O be the origin. Show that the relation $R = \{(P, Q) : P, Q \in A$ and $OP = OQ\}$ is an equivalence relation.
11. On the set S of all real numbers, define a relation $R = \{(a, b) : a \leq b\}$.
    Show that R is (i) reflexive (ii) transitive (iii) not symmetric .
12. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $R = \{(a, b) : a, b \in A$ and $b = a + 1\}$.
    Show that R is (i) not reflexive, (ii) not symmetric and (iii) not transitive.

ANSWERS (EXERCISE 1A)

1. $\text{dom}(R) = \{-1, 1, -2, 2\}$ and $\text{range}(R) = \{1, 4\}$

2. range $(R) = \{8, 27\}$

3. (i) $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$ (ii) $\text{dom}(R) = \{2, 3, 5, 7\}$
(iii) $\text{range}(R) = \{8, 27, 125, 343\}$

4. $\{3, 2, 1\}$

5. $\text{dom}(R) = \{3, 6, 9\}$ and $\text{range}(R) = \{3, 2, 1\}$

6. $\text{dom}(R) = \{-2, -1, 0, 1, 2\}$ and $\text{range}(R) = \{3, 2, 1, 0\}$

7. $\text{dom}(R) = \{2, 3, 4\}$ and $\text{range}(R) = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$

8. $\text{dom}(R) = \{1, 2, 3\}$ and $\text{range}(R) = \{6, 7, 8\}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 1A)

2. $R = \{(2, 8), (3, 27)\}$

4. $R = \{(2, 3), (4, 2), (6, 1)\}$

5. $R = \{(3, 3), (6, 2), (9, 1)\}$

6. Clearly, a is an integer such that $-3 < a < 3$.
$\therefore \quad R = \{(-2, 3), (-1, 2), (0, 1), (1, 0)(2, 1)\}$.

7. $R = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right)\right\}$

8. $R = \{(1, 6), (2, 7), (3, 8)\}$.

9. (i) $(A \subset B \text{ and } B \subset C) \Rightarrow (A \subset C)$. So, R is transitive.
(ii) Clearly, $A \subset C$ is not true. So, R is not reflexive.
(iii) If $A \subset B$ then $B \subset C$ is not true. So, R is not symmetric.

10. Let O be the origin and let P, Q, X be any three points in a plane.
Then, (i) $OP = OP$ is always true. So, R is reflextive.
(ii) $OP = OQ \Rightarrow OQ = OP$. So, R is symmetric.
(iii) $(OP = OQ \text{ and } OQ = OX) \Rightarrow (OP = OX)$. So, R transitive.
Hence, R is an equivalence relation.

11. (i) For all $a \in R$, $a \le a$ is alway true. So, R is reflexive.
(ii) For all $a, b, c \in R$, we have $(a \le b, b \le c) \Rightarrow (a \le c)$.
$\therefore$ R is transitive.
(iii) But R is not symmetrtic, as $3 \le 4$ is true, while $4 \le 3$ is not true.

12. (i) $(1, 1) \notin R$ as $1 = 1 + 1$ is not true.
(ii) $(2 = 1 + 1) \Rightarrow 1 \; R \; 2$. But $1 = 2 + 1$ is not true. So, 2 is not related to 1.
$\therefore$ R is not symmetric.
(iii) $1 \; R \; 2$ and $2 \; R \; 3$. But, 1 is not related to 3.

EXERCISE 1B

1. Define a relation on a set. What do you mean by the domain and range of a relation. Give an example.
2. Let A be the set of all triangles in a plane. Show that the relation $R = \{(\triangle_1, \triangle_2) : \triangle_1 \sim \triangle_2\}$ is an equivalence relation on A.
3. Let $R = \{(a, b) : a, b \in Z \text{ and } (a + b) \text{ is even}\}$.
   Show that R is an equivalence relation on Z.
4. Let $R = \{(a, b) : a, b \in Z \text{ and } (a - b) \text{ is divisible by } 5\}$.
   Show that R is an equivalence relation on Z.
5. Show that the relation R defined on the set $A = \{1, 2, 3, 4, 5\}$, given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. **[CBSE 2009]**
6. Show that the relation R on $N \times N$, defined by
   $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$
   is an equivalent relation. **[CBSE 2010]**
7. Let S be the set of all real numbers and let
   $R = \{(a, b) : a, b \in S \text{ and } a = \pm b\}$.
   Show that R is an equivalence relation on S.
8. Let S be the set of all points in a plane and let R be a relation in S defined by $R = \{(A, B) : d(A, B) < 2 \text{ units}\}$, where $d(A, B)$ is the distance between the points A and B.
   Show that R is reflexive and symmetric but not transitive.
9. Let S be the set of all real numbers. Show that the relation $R = \{(a, b) : a^2 + b^2 = 1\}$ is symmetric but neither reflextive nor transitive. **[CBSE 2008]**
10. Let $R = \{(a, b) : a = b^2\}$ for all $a, b \in N$.
    Show that R satisfies none of reflexivity, symmetry and transitivity.
11. Show that the relation $R = \{(a, b) : a > b\}$ on N is transitive but neither reflexive nor symmetric.
12. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$.
    Show that R is reflexive but neither symmetric nor transitive.
13. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 2)\}$. Show that R is reflexive and transitive but not symmetric.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 1B)

3. (i) Let $a \in Z$. Then, $a + a = 2a$, which is even. So, $a\ R\ a$.
   (ii) Let $a\ R\ b$. Then,
   $a\ R\ b \Rightarrow a + b$ is even
   $\Rightarrow b + a$ is even
   $\Rightarrow b\ R\ a$.

(iii) Let $a\ R\ b$ and $b\ R\ c$. Then,

$a\ R\ b$ and $b\ R\ c \Rightarrow (a+b)$ is even and $(b+c)$ is even.

$\therefore \quad (a+c) = \{(a+c)+2b\} - 2b$

$= \{(a+b)+(b+c)-2b\}$, which is even

$\Rightarrow \quad a\ R\ c$ [$\because$ each of $(a+b)$, $(b+c)$ and $2b$ is even].

4. $(a-c) = (a-b)+(b-c)$.

5. (i) $|a-a| = 0$, which is even. So, $a\ R\ a$.

(ii) $a\ R\ b \Rightarrow |a-b|$ is even

$\Rightarrow |-(a-b)|$ is even

$\Rightarrow |b-a|$ is even

$\Rightarrow b\ R\ a$.

(iii) $a\ R\ b, b\ R\ c \Rightarrow |a-b|$ is even and $|b-c|$ is even

$\Rightarrow (a-b)$ is even and $(b-c)$ is even

$\Rightarrow \{(a-b)+(b-c)\}$ is even

$\Rightarrow (a-c)$ is even

$\Rightarrow |a-c|$ is even

$\Rightarrow a\ R\ c$.

8. (i) $d(A, A) = 0 < 2$. So, $A\ R\ A$.

(ii) $A\ R\ B \Rightarrow d(A, B) < 2$

$\Rightarrow d(B, A) < 2 \quad [\because d(B, A) = d(A, B)]$

$\Rightarrow B\ R\ A$.

(iii) Consider the points $A(0, 0)$, $B(1.5, 0)$ and $C(3, 0)$ on the x-axis.

Then, $d(A, B)1.5$, $d(B, C) = 1.5$ and $d(A, C) = 3$.

$\therefore \quad A\ R\ B$ and $B\ R\ C$. But, A is not related to C.

So, R is not transitive.

9. (i) R is symmetric, since

$a\ R\ b \Rightarrow a^2 + b^2 = 1$

$\Rightarrow b^2 + a^2 = 1 \Rightarrow b\ R\ a$.

(ii) R is not reflexive, since 1 is not related to 1, as

$1^2 + 1^2 = 1$ is not true.

(iii) Clearly, $\frac{1}{2}\ R\ \frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{2}\ R\ \frac{1}{2}$.

But, $\frac{1}{2}$ is not related to $\frac{1}{2}$ as $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \neq 1$.

$\therefore \quad R$ is not transitive.

10. (i) $2 \neq 2^2 \Rightarrow 2$ is not related to 2.

(ii) $4 = 2^2 \Rightarrow 4\ R\ 2$. But $2 \neq 4^2$. So, $2\ \not{R}\ 4$.

(iii) $16\ R\ 4$, $4\ R\ 2$. But 16 is not related to 2, as $16 \neq 2^2$.

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 2), (1, 2)\}$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
2. Let $A = \{a, b, c\}$ and let $R = \{(a, a), (a, b), (b, a)\}$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
3. Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) symmetric and transitive but not reflexive
   (c) reflexive and transitive but not symmetric
   (d) an equivalence relation
4. Let S be the set of all straight lines in a plane. Let R be a relation on S defined by $a\ R\ b \Leftrightarrow a \perp b$. Then, R is
   (a) reflexive but neither symmetric nor transitive
   (b) symmetric but neither reflexive nor transitive
   (c) transitive but neither reflexive nor symmetric
   (d) an equivalence relation
5. Let S be the set of all straight lines in a plane. Let R be a relation on S defined by $a\ R\ b \Leftrightarrow a \parallel b$. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
6. Let Z be the set of all integers and let R be a relation on Z defined by $a\ R\ b \Leftrightarrow (a - b)$ is divisible by 3. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation

7. Let R be a relation on the set N of all natural numbers, defined by $a\ R\ b \Leftrightarrow a$ is a factor of b. Then, R is
   (a) reflexive and symmetric but not transitive
   (b) reflexive and transitive but not symmetric
   (c) symmetric and transitive but not reflexive
   (d) an equivalence relation
8. Let Z be the set of all integers and let R be a relation on Z defined by $a\ R\ b \Leftrightarrow a \geq b$. Then, R is
   (a) symmetric and transitive but not reflexive
   (b) reflexive and symmetric but not transitive
   (c) reflexive and transitive but not symmetric
   (d) an equivalence relation
9. Let S be the set of all real numbers and let R be a relation on S defined by $a\ R\ b \Leftrightarrow |a| \leq b$. Then, R is
   (a) reflexive but neither symmetric nor transitive
   (b) symmetric but neither reflexive nor transitive
   (c) transitive but neither reflexive nor symmetric
   (d) none of these
10. Let S be the set of all real numbers and let R be a relation on S, defined by $a\ R\ b \Leftrightarrow |a - b| \leq 1$. Then, R is
    (a) reflexive and symmetric but not transitive
    (b) reflexive and transitive but not symmetric
    (c) symmetric and transitive but not reflexive
    (d) an equivalence relation
11. Let S be the set of all real numbers and let R be a relation on S, defined by $a\ R\ b \Leftrightarrow (1 + ab) > 0$. Then, R is
    (a) reflexive and symmetric but not transitive
    (b) reflexive and transitive but not symmetric
    (c) symmetric and transitive but not reflexive
    (d) none of these
12. Let S be the set of all triangles in a plane and let R be a relation on S defined by $\Delta_1 S\ \Delta_2 \Leftrightarrow \Delta_1 \equiv \Delta_2$. Then, R is
    (a) reflexive and symmetric but not transitive
    (b) reflexive and transitive but not symmetric
    (c) symmetric and transitive but not reflexive
    (d) an equivalence relation

13. Let S be the set of all real numbers and let R be a relation on S defined by $a\ R\ b \Leftrightarrow a^2 + b^2 = 1$. Then, R is

    (a) symmetric but neither reflexive nor transitive

    (b) reflexive but neither symmetric nor transitive

    (c) transitive but neither reflexive nor symmetric

    (d) none of these

14. Let R be a relation on $N \times N$, defined by $(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$. Then, R is

    (a) reflexive and symmetric but not transitive

    (b) reflexive and transitive but not symmetric

    (c) symmetric and transitive but not reflexive

    (d) an equivalence relation

15. Let A be the set of all points in a plane and let O be the origin. Let $R = \{(P, Q) : OP = OQ\}$. Then, R is

    (a) reflexive and symmetric but not transitive

    (b) reflexive and transitive but not symmetric

    (c) symmetric and transitive but not reflexive

    (d) an equivalence relation

16. Let Q be the set of all rational numbers and $*$ be the binary operation, defined by $a * b = a + 2b$, then

    (a) $*$ is commutative but not associative

    (b) $*$ is associative but not commutative

    (c) $*$ is neither commutative nor associative

    (d) $*$ is both commutative and associative

17. Let $a * b = a + ab$ for all $a, b \in Q$. Then,

    (a) $*$ is not a binary composition

    (b) $*$ is not commutative

    (c) $*$ is commutative but not associative

    (d) $*$ is both commutative and associative

18. Let Q^+ be the set of all positive rationals. Then, the operation $*$ on Q^+ defined by $a * b = \frac{ab}{2}$ for all $a, b \in Q^+$ is

    (a) commutative but not associative

    (b) associative but not commutative

    (c) neither commutative nor associative

    (d) both commutative and associative

19. Let Z be the set of all integers and let $a * b = a - b + ab$. Then, $*$ is
    (a) commutative but not associative
    (b) associative but not commutative
    (c) neither commutative nor associative
    (d) both commutative and associative
20. Let Z be the set of all integers. Then, the operation $*$ on Z defined by $a * b = a + b - ab$ is
    (a) commutative but not associative
    (b) associative but not commutative
    (c) neither commutative nor associative
    (d) both commutative and associative
21. Let Z^+ be the set of all positive integers. Then, the operation $*$ on Z^+ defined by $a * b = a^b$ is
    (a) commutative but not associative
    (b) associative but not commutative
    (c) neither commutative nor associative
    (d) both commutative and associative
22. Define $*$ on $Q - \{-1\}$ by $a * a + b + ab$. Then, $*$ on $Q - \{-1\}$ is
    (a) commutative but not associative
    (b) associative but not commutative
    (c) neither commutative nor associative
    (d) both commutative and associative

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) **2.** (c) **3.** (a) **4.** (b) **5.** (d) **6.** (d) **7.** (b) **8** (c) **9.** (c) **10.** (a)
11. (a) **12.** (d) **13.** (a) **14.** (d) **15.** (d) **16.** (c) **17.** (b) **18.** (d) **19.** (c) **20.** (d)
21. (c) **22.** (d)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. R is reflexive and transitive but not symmetric.

2. R is symmetric and transitive but not reflexive.

3. $(1, 2) \in R$ and $(2, 3) \in R$. But, $(1, 3) \notin R$. So, R is not transitive.

4. $a \perp a$ is not true. So, R is not reflexive.
$a \perp b$ and $b \perp c$ do not imply $a \perp c$. So, R is not transitive.
But, $a \perp b \Rightarrow b \perp a$ is always true.

9. (i) $|-3| \leq -3$ is not true. So, R is not reflexive.

(ii) $|-1| \leq 1 \Rightarrow (-1)\ R\ 1$. But $|1| \leq -1$ is not true.

$\therefore$ R is not symmetric.

(iii) $a\ R\ b, b\ R\ c \Rightarrow |a| \leq b$ and $|b| \leq c \Rightarrow |a| \leq c$.

$\therefore$ R is transitive.

10. (i) $|a - a| = 0 \leq 1$ is always true.

(ii) $a\ R\ b \Rightarrow |a - b| \leq 1 \Rightarrow |-(a - b)| \leq 1 \Rightarrow |b - a| \leq 1 \Rightarrow b\ R\ a$.

(iii) $2\ R\ 1$ and $1\ R\ \frac{1}{2}$.

But, 2 is not related to $\frac{1}{2}$. So, R is not transitive.

11. (i) $a\ R\ a$, since $(1 + a^2) > 0$.

(ii) $a\ R\ b \Rightarrow (1 + ab) > 0 \Rightarrow (1 + ba) > 0 \Rightarrow b\ R\ a$.

(iii) Let $a = \frac{-1}{2}, b = \frac{1}{2}$ and $c = R$.

Then, $a\ R\ b$ and $b\ R\ c$. But, a is not related to c.

13. (i) $(1^2 + 1^2) \neq 1$. So, $1\ R\ 1$ is not true.

(ii) $a\ R\ b \Rightarrow a^2 + b^2 = 1 \Rightarrow b^2 + a^2 = 1 \Rightarrow b\ R\ a$.

(iii) $1\ R\ 0$ and $0\ R\ 1$. But, 1 is not related to 1.